Phys 410
Spring 2013
Lecture #30 Summary
8 April, 2013

The generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, constitute a set of $2n$ quantities that span **phase space**. The instantaneous state of the entire system is summarized as a single mathematical point in this phase space. Call this point $\vec{z} = (\vec{q}, \vec{p})$, where $\vec{q} = (q_1, \dots q_n)$ is an ordered list of the n generalized coordinates, and $\vec{p} = (p_1, \dots p_n)$ is the list of n conjugate momenta. Hamiltonian's equations describe how this point moves in phase space – in other words it describes the trajectory of the phase point. You can see this by taking the time derivative of $\vec{z}$ as $\dot{\vec{z}} = (\dot{\vec{q}}, \dot{\vec{p}})$, and noting the fact that : $\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i} = f_i(\vec{q}, \vec{p})$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i = g_i(\vec{q}, \vec{p})$, $i = 1, \dots, n$, where the vector functions $\vec{f}(\vec{q}, \vec{p})$, $\vec{g}(\vec{q}, \vec{p})$ summarize the derivatives of the Hamiltonian with respect to the coordinates and momenta. Thus we have the first order differential equation for the trajectory of the phase point: $\dot{\vec{z}} = \left(\vec{f}(\vec{q}, \vec{p}), \vec{g}(\vec{q}, \vec{p})\right) = \left(\vec{f}(\vec{z}), \vec{g}(\vec{z})\right)$. This is a deterministic equation for the evolution of the phase point. It shows that two trajectories that arise from two different initial conditions can never cross, because otherwise there would be two different trajectories arising from the same equation with the same instantaneous value of $\vec{z}$, contrary to the deterministic nature of the phase point evolution equation.

We considered the $2n = 2$ −dimensional phase space of a $n = 1$ one-dimensional harmonic oscillator. The trajectory of the phase point is an ellipse in the (x, p) phase plane. We also briefly considered the case of an object falling under the influence of gravity in one dimension. In this case the phase point trajectories are parabolas in phase space: $x = \frac{p^2}{2m^2 g} - \frac{E}{mg}$, where E is the total mechanical energy.